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Drop induced impact response of a printed wiring board

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Abstract

Analytical solution is given to the Duffing equation in its free response induced by the initial velocity. Maximum deflection and acceleration are analyzed for its behavior in both hard and soft springs. The Duffing equation is derived for the transient response of a laminated printed wiring board (PWB) as a hard spring system. Effect of the system parameters on the nonlinear response is analyzed for the PWB. Results are generated to characterize the response behavior with respect to the modal parameters and structure design of PWB. It is found that the maximum deflection is almost linearly proportional to the initial velocity induced by the impact momentum.

Keywords: Laminate; Duffing's equation; Printed wiring board; Impact; Nonlinear dynamics

1. Introduction

Dynamic response of nonlinear system governed by the Duffing equation has been studied for different initial and loading conditions. Analytical solutions have been derived for the system free response induced by the initial deflection by Hsu (1960), as well as response under constant load by Suhir (1992, 1995). Perurbation method and numerical computation has been used for the forced harmonic response (e.g. by Stoker, 1950). The response induced by the initial velocity has not been completely derived in the known literature (Davis, 1962).

In this study, analytical solution to the free response of the Duffing equation induced by the initial velocity is obtained in the form of an elliptic function. Both hard and soft spring systems of the Duffing equation are analyzed for its response characters. For the hard spring system, special application to the transient response of a printed wiring board (PWB) made of symmetric isotropic laminates is studied. The response induced by the initial velocity represents the impact reaction of a flatly dropped PWB with constrained boundaries, e.g. simply supported. The initial velocity is induced from the momentum conservation. The PWB governing equation of motion is based on the nonlinear dynamics analysis of the laminated composite in modal approach (He, 2000).

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2. Free response due to initial velocity

The solution to the transient response of the homogeneous Duffing equation in its modal analysis format

$$\ddot{W}_{mn}(t) + \omega_{mn}^2 W_{mn}(t) + r_{mn} W_{mn}^3(t) = 0 \tag{1}$$

subjected to the condition of:

$$W_{mn}(0) = 0, \qquad W_{mn,t}(0) = V_{mn}^0 \tag{2}$$

can be assumed in the form of:

$$W_{mn}(t) = \frac{V_{mn}^0}{\sigma_{mn}} sn(u, k) \quad \text{with } u = \sigma_{mn} t \tag{3}$$

Here the index m and n represent the two dimensional modal parameters, such as that used in the PWB dynamics analysis for the deformation mode. The modal solution for the out-of-plane deformation of the laminate is assumed as:

$$w_{mn}(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{mn}(t) \sin(\alpha_m x) \sin(\beta_n y) \tag{4}$$

where

$$\alpha_m = \frac{m\pi}{a}, \qquad \beta_n = \frac{n\pi}{b} \tag{5}$$

Its free response magnitude $W_{mn}(t)$ in large deflection is governed by Eq. (1).

The function $sn(u, k)$ is the Jocobi elliptic function of the first kind, with module k. Here u and k are associated with modal response. It should be referred to as u_{mn} and k_{mn}. For simplicity, the notion with m and n are not used. In the sequel, the elliptic functions $cn(u, k)$ and $dn(u, k)$ will also occur.

Using the results of differentiation of Eq. (3), Eq. (1) can be expressed as:

$$-\sigma_{mn}^4[1 + k^2 - 2k^2 sn^2(\sigma_{mn} t, k)] + \omega_{mn}^2 \sigma_{mn}^2 + r_{mn}[V_{mn}^0]^2 sn^2(u, t) = 0 \tag{6}$$

from which the parameters of $sn(u, k)$ function are found to satisfy:

$$\sigma_{mn}^2 = \frac{\omega_{mn}^2}{1 + k^2}, \qquad k^2 = -\frac{r_{mn}[V_{mn}^0]^2}{2\sigma_{mn}^4}, \qquad \left(\frac{k}{1 + k^2}\right)^2 = -\frac{r_{mn}[V_{mn}^0]^2}{2\omega_{mn}^4} \tag{7}$$

2.1. Hard spring, $r_{mn} > 0$

Define

$$\gamma = \sqrt{\frac{r_{mn}[V_{mn}^0]^2}{2\omega_{mn}^4}}, \quad \gamma > 0 \tag{8}$$

then

$$\left(\frac{k}{1 + k^2}\right) = \mathrm{i}\gamma \tag{9}$$

Let

$$k = \mathrm{i}\kappa \quad \text{hence } \left(\frac{\mathrm{i}\kappa}{1 - \kappa^2}\right) = \mathrm{i}\gamma \tag{10}$$

Here κ is real, k is imaginary. It can be solved that:

$$\kappa_1 = \frac{-1+\sqrt{1+4\gamma^2}}{2\gamma} > 0, \qquad \kappa_2 = \frac{-1-\sqrt{1+4\gamma^2}}{2\gamma} < 0 \tag{11}$$

κ_1 and κ_2 satisfy:

$$\kappa_1\kappa_2 = -1, \text{ or } k_1k_2 = 1. \quad \text{As } \gamma \to 0, \ \kappa_1 \to 0, \ \kappa_2 \to -\infty \tag{12}$$

From Eqs. (7) and (12):

$$\sigma_{mn}^2 = \frac{\omega_{mn}^2}{1-\kappa^2} = \frac{\gamma\omega_{mn}^2}{\kappa}, \qquad \sigma_{mn\_1} = \omega_{mn}\sqrt{\frac{\gamma}{\kappa_1}}, \qquad \sigma_{mn\_2} = \omega_{mn}\sqrt{\frac{\gamma}{\kappa_2}} \tag{13}$$

$$\sigma_{mn\_1} = \frac{\omega_{mn}}{\sqrt{2}}\sqrt{\sqrt{1+4\gamma^2}+1} > 0, \quad \text{a real constant} \tag{14a}$$

$$\sigma_{mn\_2} = \mathrm{i}\frac{\omega_{mn}}{\sqrt{2}}\sqrt{\sqrt{1+4\gamma^2}-1}, \quad \text{a purely imaginary constant} \tag{14b}$$

σ_{mn_1}, σ_{mn_2} has the following relations:

$$\sigma_{mn\_1} = \mathrm{i}\kappa_2\sigma_{mn\_2} = k_2\sigma_{mn\_2}, \qquad \sigma_{mn\_2} = \mathrm{i}\kappa_1\sigma_{mn\_1} = k_1\sigma_{mn\_1} \tag{15}$$

The two solutions can be written as:

$$W_{mn\_1}(t) = \frac{V_{mn}^0}{\sigma_{mn\_1}} sn(\sigma_{mn\_1}t, k_1) = \frac{V_{mn}^0}{\sigma_{mn\_1}} sn(\sigma_{mn\_1}t, \mathrm{i}\kappa_1) \tag{16}$$

$$W_{mn\_2}(t) = \frac{V_{mn}^0}{\sigma_{mn\_2}} sn(\sigma_{mn\_2}t, k_2) = \frac{V_{mn}^0}{\sigma_{mn\_2}} sn(\sigma_{mn\_2}t, \mathrm{i}\kappa_2) \tag{17}$$

Using Eq. (15) for σ_{mn_1} and σ_{mn_2}, Eq. (12) for k_1 and k_2, from the following transformation (Bateman and Erdelyi, 1955):

$$sn\left(k_2u, \frac{1}{k_2}\right) = k_2 sn(u, k_2) \tag{18}$$

which holds for k either real or imaginary, the above two solutions are found identically the same for the hard spring, i.e.

$$W_{mn\_1}(t) = \frac{V_{mn}^0}{k_2\sigma_{mn\_2}} sn\left(k_2\sigma_{mn\_2}t, \frac{1}{k_2}\right) = W_{mn\_2}(t) \tag{19}$$

Since k is purely imaginary, as given by Eq. (12), the solution can be further simplified into the form of a real function based on the transformation of (Bateman and Erdelyi, 1955):

$$sn(u, \mathrm{i}\kappa) = \frac{1}{\sqrt{1+\kappa^2}} sd\left(u\sqrt{1+\kappa^2}, \frac{\kappa}{\sqrt{1+\kappa^2}}\right) \quad \text{where } sd(u,\kappa) = \frac{sn(u,\kappa)}{dn(u,\kappa)} \tag{20}$$

Therefore, the solution for a hard spring can be expressed as:

$$W_{mn\_1}(t) = \frac{V_{mn}^0}{\sigma_{mn\_1}} \frac{1}{\sqrt{1+\kappa_1^2}} sd\left(u\sqrt{1+\kappa_1^2}t, \frac{\kappa_1}{\sqrt{1+\kappa_1^2}}\right) \tag{21}$$

γ, κ_1 and σ_{mn_1} are defined in Eqs. (8), (11) and (14a), respectively.

2.2. Soft spring, $r_{mn} < 0$

Let

$$\beta = \sqrt{-\frac{r_{mn}[V_{mn}^0]^2}{2\omega_{mn}^4}} > 0 \quad \text{and} \quad \left(\frac{k}{1+k^2}\right) = \beta \tag{22}$$

Then

$$k_1 = \frac{1+\sqrt{1-4\beta^2}}{2\beta}, \qquad k_2 = \frac{1-\sqrt{1-4\beta^2}}{2\beta} \tag{23}$$

and

$$k_1 k_2 = 1 \tag{24}$$

As $\beta \to 0$, $k_1 \to \infty$, $k_2 \to 0$. When $0 < \beta \leqslant 1/2$, k_1 and k_2 are real and positive. Otherwise, k_1 and k_2 are complex. In this case, there are no available transformations for complex k to yield an analytical solution in real function.

For the case of real k_1 and k_2, from Eqs. (7) and (22):

$$\sigma_{mn}^2 = \frac{\omega_{mn}^2}{1+k^2} = \frac{\beta\omega_{mn}^2}{k} \tag{25}$$

Then

$$\sigma_{mn\_1} = \omega_{mn}\sqrt{\frac{\beta}{k_1}}, \qquad \sigma_{mn\_2} = \omega_{mn}\sqrt{\frac{\beta}{k_2}} \tag{26}$$

$$\sigma_{mn\_1} = \frac{\omega_{mn}}{\sqrt{2}}\sqrt{1-\sqrt{1-4\beta^2}}, \qquad \sigma_{mn\_2} = \frac{\omega_{mn}}{\sqrt{2}}\sqrt{1+\sqrt{1-4\beta^2}} \tag{27}$$

Both σ_{mn_1} and σ_{mn_2} are real. From Eqs. (25) and (27):

$$\sigma_{mn\_1} = k_2\sigma_{mn\_2}, \qquad \sigma_{mn\_2} = k_1\sigma_{mn\_1} \tag{28}$$

Therefore,

$$W_{mn\_1}(t) = \frac{V_{mn}^0}{\sigma_{mn\_1}} sn(\sigma_{mn\_1}t, k_1) \tag{29}$$

$$W_{mn\_2}(t) = \frac{V_{mn}^0}{\sigma_{mn\_2}} sn(\sigma_{mn\_2}t, k_2) \tag{30}$$

From the transformation Eq. (18), it can be verified that the two solutions are identical again, i.e.

$$W_{mn\_1}(t) = W_{mn\_2}(t) \tag{31}$$

The parameters β, k_1 and σ_{mn_1} for $W_{mn\_1}(t)$ are given by Eqs. (22), (23) and (26), respectively.

3. Response characterization

By integrating Eq. (1), we obtain

$$\tfrac{1}{2}\dot{W}_{mn}^2 + \tfrac{1}{2}\omega_{mn}^2 W_{mn}^2 + \tfrac{1}{4} r_{mn} W_{mn}^4 = C \tag{32}$$

for the given initial velocity in Eq. (2), then, Eq. (32) becomes:

$$\dot{W}_{mn}^2 + \omega_{mn}^2 W_{mn}^2 + \tfrac{1}{2} r_{mn} W_{mn}^4 = (V_{mn}^0)^2 \tag{33}$$

Differentiate Eq. (1)

$$\dddot{W}_{mn} + \omega_{mn}^2 \dot{W}_{mn} + 3 r_{mn} W_{mn}^2 \dot{W}_{mn} = 0 \tag{34}$$

Eqs. (33) and (34) are to be used for characterizing the maximum response of the nonlinear system behavior.

At $\dot{W}_{mn} = 0$, Eq. (33) yields:

$$W_{\substack{mn\_\max \\ mn\_\min}} = \pm \sqrt{\frac{\sqrt{\omega_{mn}^4 + 2 r_{mn} (V_{mn}^0)^2} - \omega_{mn}^2}{r_{mn}}} \tag{35}$$

Note that when $r_{mn} = 0$, representing linear system, the maximum deflection should be determined from the linear analysis as:

$$W_{mn\_\max}^{\mathrm{L}} = \frac{V_{mn}^0}{\omega_{mn}} \tag{36}$$

For the given maximum deflection of the nonlinear system, the acceleration can be determined from Eq. (1) as:

$$\ddot{W}_{\substack{mn\_\max \\ mn\_\min}} = -W_{\substack{mn\_\min \\ mn\_\max}} \left(\omega_{mn}^2 + r_{mn} W^2_{\substack{mn\_\min \\ mn\_\max}} \right) \tag{37}$$

The above equation corresponds to the condition of $\dddot{W}_{mn} = 0$. This is because at $\dddot{W}_{mn} = 0$, Eq. (34) yields:

$$\dot{W}_{mn} (\omega_{mn}^2 + 3 r_{mn} W_{mn}^2) = 0 \tag{38}$$

$$\text{for} \quad r_{mn} > 0, \quad \text{Eq. (38) means} \quad \dot{W}_{mn} = 0 \tag{39}$$

Therefore, for a hard spring, maximum acceleration and minimum deflection are obtained at the same time.

The maximum velocity for the hard spring can be obtained from Eq. (1). When $\ddot{W}_{mn} = 0$,

$$\omega_{mn}^2 W_{mn} + r_{mn} W_{mn}^3 = 0 \tag{40}$$

since

$$\omega_{mn}^2 W_{mn} + r_{mn} W_{mn}^3 \neq 0, \quad \text{then } W_{mn} = 0 \tag{41}$$

From Eq. (33), this means that $\dot{W}_{mn\_\max} = V_{mn}^0$, i.e. the initial velocity is the maximum velocity for the hard spring system.

For the maximum acceleration of the soft spring, at $\dddot{W}_{mn} = 0$, from Eq. (34):

$$\dot{W}_{mn} = 0 \quad \text{or} \quad (\omega_{mn}^2 + 3 r_{mn} W_{mn}^2) = 0 \tag{42}$$

there are three cases that can be identified.

$$\text{(a)}\quad \dot{W}_{mn} = 0, \qquad (\omega_{mn}^2 + 3r_{mn}W_{mn}^2) \neq 0 \tag{43}$$

It leads to the maximum deflection expressed in Eq. (35) and the maximum acceleration in Eq. (37). The case is identical to the hard spring.

$$\text{(b)}\quad \dot{W}_{mn} \neq 0 \quad \text{and} \quad (\omega_{mn}^2 + 3r_{mn}W_{mn}^2) = 0 \tag{44}$$

$$W_{\substack{mn\_\max \\ mn\_\min}} = \pm\omega_{mn}\left(-\frac{1}{3r_{mn}}\right)^{1/2} \tag{45}$$

The maximum acceleration is determined from Eq. (1):

$$\ddot{W}_{\substack{mn\_\max \\ mn\_\min}} = \pm\frac{2}{3}\omega_{mn}^3\left(-\frac{1}{3r_{mn}}\right)^{1/2} \tag{46}$$

The magnitude of the response in cases (a) and (b) is compared and found that when

$$V_{mn}^0 \geqslant \omega_{mn}^2\sqrt{-\frac{5}{18r_{mn}}} \tag{47}$$

The maximum deflection in case (a) is equal or higher than that in case (b), so does the acceleration.

$$\text{(c)}\quad \dot{W}_{mn} = 0, \qquad (\omega_{mn}^2 + 3r_{mn}W_{mn}^2) = 0 \tag{48}$$

From Eqs. (33) and (45), at velocity $\dot{W}_{mn} = 0$,

$$\dot{W}_{mn} = \pm\sqrt{\frac{5}{18}\left(\frac{\omega_{mn}^4}{r_{mn}}\right) + (V_{mn}^0)^2} = 0, \qquad V_{mn}^0 = \sqrt{-\frac{5}{18}\left(\frac{\omega_{mn}^4}{r_{mn}}\right)} \tag{49}$$

This means that the condition (c) is valid only at the particular initial velocity as given by Eq. (49). Otherwise it would not occur. In case (c), the acceleration is given the same as that in case (b). The response of the soft spring is a combination of each case.

4. PWB response as a hard spring system

A simply supported laminated PWB is analyzed. Its out-of-plane motion is in the form of Eq. (4), with boundary conditions:

$$\begin{gathered} u_0(0,y,t) = 0, \quad u_0(a,y,t) = 0, \quad v_0(x,0,t) = 0, \quad v_0(x,b,t) = 0, \quad w(x,0,t) = 0, \quad w(x,b,t) = 0, \\ w(0,y,t) = 0, \quad w(a,y,t) = 0 \quad \text{and} \quad N_{xy}(x,y,t) = 0, \quad F_{,xy}(x,0,t) = 0 \quad \text{on all the boundaries.} \end{gathered} \tag{50}$$

The governing equation for the PWB transient response has been derived based on the Von-Karman nonlinear strain field for the large deflection in the Duffing equation, with the following coefficients (He, 2000):

$$r_{mn} = \frac{1}{16\tilde{I}}\left[4A_{12}\alpha_m^2\beta_n^2 + \frac{3A_{11}^2 - A_{12}^2}{A_{11}}(\alpha_m^4 + \beta_n^4)\right] \tag{51}$$

$$\omega_{mn}^2 = \frac{D_{11}\lfloor\alpha_m^2 + \beta_n^2\rfloor}{\tilde{I}} \tag{52}$$

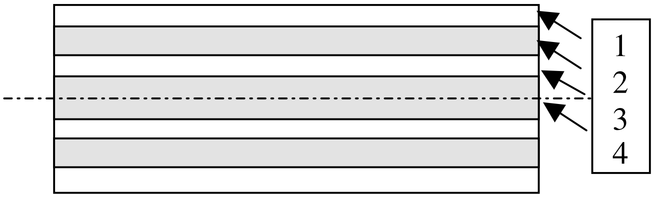

Fig. 1. PWB in isotropic laminate (length = 154 mm, width = 154 mm, height = 1.53 mm).

Table 1
Material properties of the PWB laminate

Layer	Material	Specific weight (g/cm^3)	Young's module (MPa)	Poission's ratio	Thickness (mm)
1	Cu/FR-4	7.599	2.98×10^4	0.205	0.030
2	FR-4	1.200	2.0×10^4	0.190	0.200
3	Cu	8.310	1.18×10^5	0.340	0.035
4	FR-4	1.200	2.0×10^4	0.190	1.000

Table 2
Modal parameters for the Duffing equation

Mode		Simple support	
M	N	ω_{mn} (Hz)	r_{mn} ($mm^{-2}\,s^{-2}$)
1	1	1588	1,246,924
1	2	3970	9,851,242
8	8	101,093	5,049,969,553

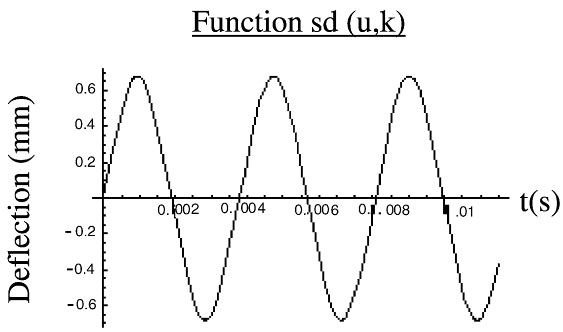

Fig. 2. Mathematica deflection result with $v_{11}^0 = 1$ m/s.

where $\tilde{I}$ is the inertia of the system. The Duffing equation for PWB is a hard spring oscillator due to the laminate stiffness and rigidity coefficients $A_{ij} > 0$, $D_{ij} > 0$, hence $r_{mn} > 0$. A plastic PWB in symmetric lay-up of isotropic laminates is shown in Fig. 1. It is made of copper and FR-4 laminae. Dimensions are $a \times b = 154 \times 154$ mm in length and width, total thickness is $h = 1.53$ mm. The material properties of each lamina are given in Table 1. Some selective modal parameters for the Duffing equation are listed in Table 2.

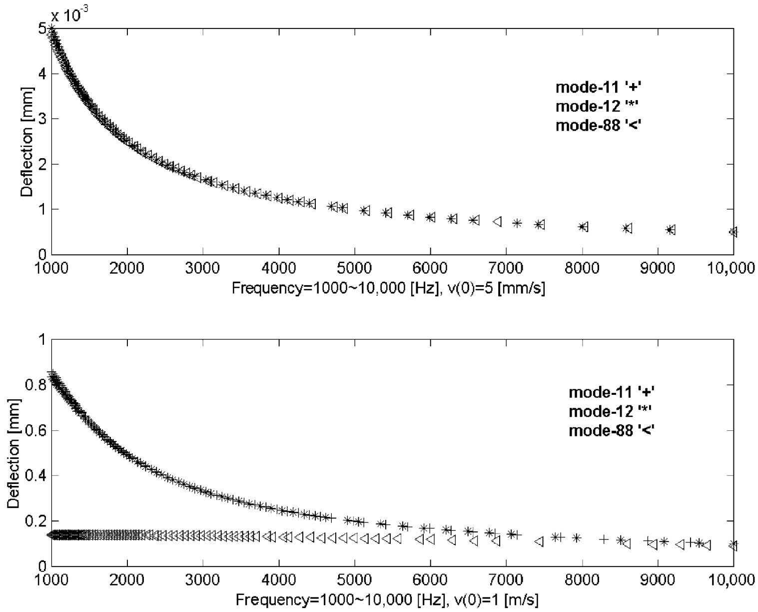

Fig. 3. Maximum deflection of hard spring with stiffness and frequency variation.

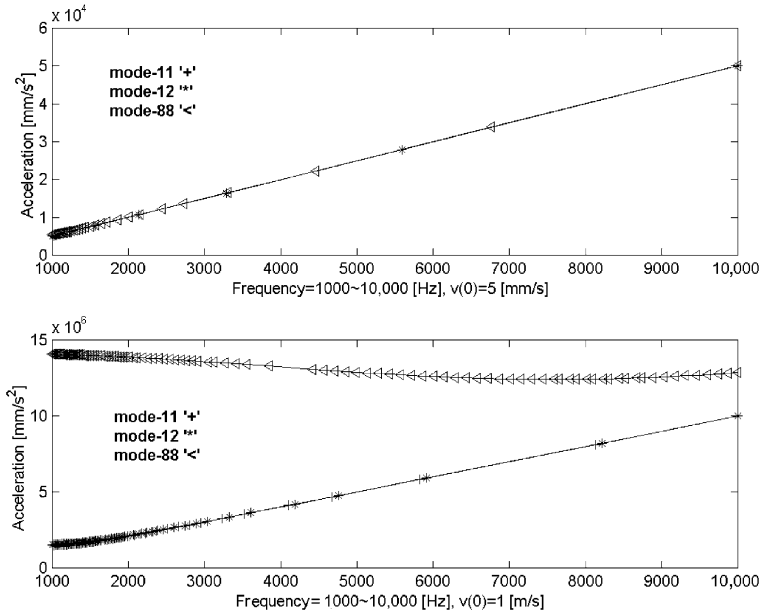

Fig. 4. Maximum acceleration of hard spring with stiffness and frequency variation.

The PWB response in the fundamental mode with initial velocity $v_{11}^0 = 1$ m/s is shown in Fig. 2. It is generated from *Mathematica* for the solution expressed in Eq. (21).

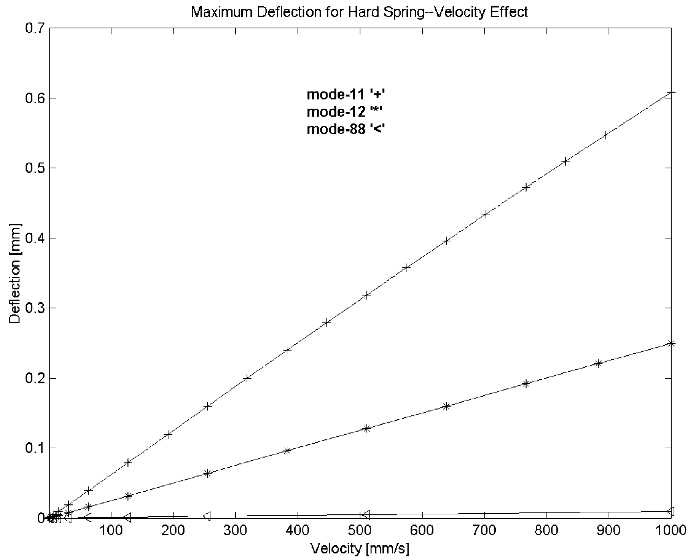

Fig. 5. Maximum deflection for hard spring with initial velocity variation.

The PWB response character in relation to the parameter ω_{mn} and r_{mn} can be studied from Eqs. (35) and (37). Plots of the maximum deflection and acceleration with respect to the frequency in the range of 1000–10,000 Hz are shown in Figs. 3 and 4, respectively. The stiffness of each curve is chosen to be the modal parameters listed in Table 2. The initial velocity chosen for the plot is 5 and 1000 mm/s, respectively, in the two subplots as shown in each figure.

It is found that increased frequency reduces the maximum deflection. The maximum deflection at $v_{mn}^0 = 5$ mm/s and frequency 1000 Hz is about $5 \times 10^{(-3)}$ mm, while the maximum deflection at 1000 mm/s is about 0.83 mm. Also noted from Fig. 3 is that the three deflection curves with different stiffness converge at $v_{mn}^0 = 5$ mm/s. This is an indication of the negligible effect of the stiffness to the deflection induced by the initial velocity. The convergence is also observed for mode-11 and mode-12 at velocity $v_{mn}^0 = 1000$ mm/s. For mode-88, the deflection is significantly reduced to about 0.16 mm, and almost invariant with respect to the frequency at higher initial velocity. Since higher frequency and stiffness are associated with higher modes, the insignificant deflection of high mode and the decline of lower mode deflection curve with respect to the frequency is consistent with the modal analysis results for the nonlinear system, i.e. higher modal response yields deflection that can be neglected.

The acceleration increases with respect to the frequency linearly as seen in Fig. 4. Convergence of the curves with different stiffness is also observed for different modes at both velocities. High acceleration is observed for mode-88 at higher velocity of $v_{mn}^0 = 1000$ mm/s, which makes it divert from the other curves.

The effect of the initial velocity v_{mn}^0 on the maximum deflection and the acceleration is shown in Figs. 5 and 6, respectively. Distinction is noticed between curves with different stiffness and frequency. The parameters for the three curves are chosen as listed in Table 2. It is found that increased initial velocity, as of the consequence of high momentum impact, leads to significant increase of deflection in the lower modes, and has negligible effect for the higher mode. At $v_{mn}^0 = 1000$ mm/s, the maximum deflection for the first mode is about 0.63 mm, which agrees with that shown in Fig. 3. High initial velocity leads to higher acceleration for the high modes. Almost linear variation of both deflection and acceleration with respect to the initial velocity is observed.

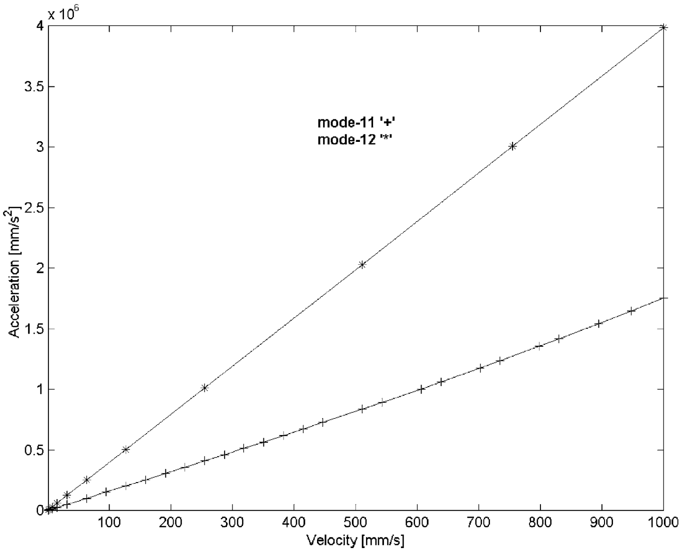

Fig. 6. Maximum acceleration of hard spring with initial velocity variation.

5. Discussion

The maximum deflection of the Duffing system given by Eqs. (35) and (37) can be expressed, alternatively, in series. From Eq. (35), the maximum deflection can be in the form of:

$$W_{\substack{mn\_\max \\ mn\_\min}} = \pm \frac{\omega_{mn}}{\sqrt{r_{mn}}} \left[\left(1 + \frac{2r_{mn}(V^0_{mn})^2}{\omega^4_{mn}} \right)^{1/2} - 1 \right]^{1/2} \tag{53}$$

Define

$$\alpha_{mn} = \frac{\omega^4_{mn}}{2r_{mn}} \tag{54}$$

$$W_{\substack{mn\_\max \\ mn\_\min}} = \pm \frac{\omega_{mn}}{\sqrt{r_{mn}}} \left[\left(1 + \frac{(V^0_{mn})^2}{\alpha_{mn}} \right)^{1/2} - 1 \right]^{1/2} \tag{55}$$

From Eq. (37),

$$\frac{\ddot{W}_{mn\_\max}}{W_{mn\_\min}} = \frac{\ddot{W}_{mn\_\min}}{W_{mn\_\max}} = -(\omega^2_{mn} + r_{mn} W^2_{mn\_\max}) \tag{56}$$

Eq. (56) indicates that both deflection and acceleration contains α_{mn} as a common factor. Based on Table 2, it can be verified that $\alpha_{mn} > 10^6$. Specifically, the ratios for different modes are:

$$\alpha_{11} = 2.5534\text{e}+006, \qquad \alpha_{12} = 1.2618\text{e}+007, \qquad \alpha_{88} = 1.0341\text{e}+010 \tag{57}$$

The magnitude in Eq. (57) means that frequency has a dominant effect on the response. The effect of α_{mn} on the deflection can be accessed approximately by its series expression, i.e.

$$W_{\substack{mn\_\max\\mn\_\min}} = \pm\frac{V_{mn}^0}{\omega_{mn}}\left[1 - \frac{(V_{mn}^0)^2}{8\alpha_{mn}} + \frac{(V_{mn}^0)^4}{16\alpha_{mn}^2} + \cdots\right] \tag{58}$$

It is seen that it contains the power series of the term $(\alpha_{mn})^{-1}$. These terms are trivial based on Eq. (57). It is significant only when high initial velocity is present, i.e. $v_{11}^0 > 1$ m/s. For $v_{11}^0 \leqslant 1$ m/s, Eq. (61) can be approximated as:

$$W_{\substack{mn\_\max\\mn\_\min}} \leqslant \pm 0.96\frac{V_{mn}^0}{\omega_{mn}} \tag{59}$$

Therefore, the deflection varies inversely with the frequency and almost linearly with the initial velocity, which is supported by Figs. 3 and 5, respectively. This relationship exists on the condition of the ratio defined by Eq. (57). Similar analysis for the acceleration from Eq. (37) yields,

$$\ddot{W}_{mn\_\min} \cong 0.96V_{mn}^0\omega_{mn}\left(1 + 0.96\frac{(V_{mn}^0)^2}{2\alpha_{mn}}\right) \tag{60}$$

For the fundamental mode at $v_{11}^0 \leqslant 1$ m/s, the response can be approximated as:

$$\ddot{W}_{11\_\min} = V_{11}^0\omega_{11}\left(1 + 0.19(V_{11}^0)^2\right) \tag{61}$$

For mode-12, the acceleration is:

$$\ddot{W}_{12\_\min} = V_{12}^0\omega_{12}\left(1 + 0.038(V_{12}^0)^2\right) \cong V_{12}^0\omega_{12} \tag{62}$$

Hence, the acceleration varies almost linearly with the frequency and the initial velocity for higher modes. For the fundamental mode, this linear variation is a close approximation when the velocity is insignificant. The linear variation assumes either that the stiffness is fixed, or the ratio α_{mn} is in the range defined by Eq. (57). It agrees with the results shown in Figs. 4 and 6, respectively.

It should be noted that frequency and stiffness of the Duffing equation in Eq. (54) are defined by the material properties and the structure of the laminated PWB. The flexural rigidity of the laminate, D_{11}, and the stiffness, A_{11}, are given by:

$$D_{11} = \sum_{n=1}^{k}\frac{E_k h_k^3}{12(1-v_k)}, \qquad A_{11} = \sum_{n=1}^{k}\frac{E_k h_k}{12(1-v_k)}, \qquad A_{12} = \sum_{n=1}^{k}\frac{v_k E_k h_k}{12(1-v_k)} \tag{63}$$

Hence, material and the lamina thickness can be assessed for its effect on the system behavior. For example, by using lamina with increased thickness, both A_{11} and D_{11} can be increased. Alternatively, material with higher E_k can be used. Both approaches can increase the nonlinear system frequency ω_{mn} and stiffness r_{mn}, as seen from Eqs. (51) and (52). Therefore deflection can be reduced, hence reduced stresses caused by the deflection.

6. Conclusion

Analytical solution is obtained for free response of the Duffing equation induced by the initial velocity. The effect of the system parameters and the initial velocity on the maximum response is studied to characterize the nonlinear system behavior. Hard spring system is analyzed for a PWB's transient response. It is found that for the given PWB parameter, stiffness r_{mn} is not sensitive to the maximum deflection induced by the initial velocity. The increased system frequency leads to reduced deflection for both hard and soft springs in an inverse function. Deflection and acceleration are approximately linear function of the initial velocity. Results suggest that the fundamental mode response for the PWB is essential.

Optimization of the PWB behavior can be made based on the analytical expression of the system parameters. The analytical result obtained can predict deflection; hence the stresses induced in the large deflection of the thin laminated PWB during impact. Deflection can be effectively reduced by adjusting the material properties, structure thickness as well as the initial impact momentum.

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